A tight entropy-power uncertainty relation

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The uncertainty principle lies at the heart of quantum physics. It exhibits one of the key divergences between a classical and a quantum system. The original uncertainty relation, due to Heisenberg [1] and Kennard [2], relies on the variances of $\hat{x}$ and $\hat{p}$. It is written as

$$\sigma_x^2\,\sigma_p^2 \geq (\hbar/2)^2. \qquad (1)$$

Relation (1) is invariant under (x,p)-displacements in phase space and it is saturated by all pure Gaussian states provided that they are squeezed in the x or p direction only. More precisely, the Heisenberg relation is saturated for pure Gaussian states provided the principal axes of the covariance matrix γ are aligned with the x- and p-axes, namely $\sigma_{xp} = 0$.

The Heisenberg relation was improved by Schrödinger and Robertson [3, 4] by taking into account the covariance σ_{xp} (through the covariance matrix). For two canonically-conjugate variables $\hat{x}$ and $\hat{p}$, it is written as

$$|\gamma| \geq (\hbar/2)^2. \qquad (2)$$

Relation (2) has the advantage that it is now saturated by all pure Gaussian states, regardless of the orientation of the principal axes of the covariance matrix. Thus, this uncertainty relation is invariant under all Gaussian unitary transformations (displacements and symplectic transformations).

A different kind of uncertainty relations, originated by Bialynicki-Birula and Mycielski [5], relies on Shannon differential entropies $h(x)$ and $h(p)$ instead of variances as a measure of uncertainty. It is expressed as

$$h(x) + h(p) \geq \ln(\pi e \hbar). \qquad (3)$$

Interestingly, this relation can be written in terms of entropy powers defined as $N_x = (2\pi e)^{-1}\, \exp\{2\,h(x)\} \leq \sigma_x^2$ and $N_p = (2\pi e)^{-1}\, \exp\{2\,h(p)\} \leq \sigma_p^2$ so that eq. (3) becomes

$$N_x\, N_p \geq (\hbar/2)^2\,, \qquad (4)$$

which is what we call an *entropy-power uncertainty relation* for a pair of canonically-conjugate variables: it closely resembles the Heisenberg relation (1), but with entropy powers instead of variances. Furthermore, it is now obvious to see that this relation implies the Heisenberg one. However, it is not invariant under all Gaussian unitaries, unlike eq. (2).

In this contribution, we prove by variational calculus (under a reasonable assumption) a tighter form of entropy-power uncertainty relation that is extended to rotated variables by taking correlations into account [6]. It is is written as

$$h(x) + h(p) - \frac{1}{2}\ln\left(\frac{\sigma_x^2\sigma_p^2}{|\gamma|}\right) \geq \ln(\pi e \hbar). \qquad (5)$$

in terms of differential entropies or equivalently as

$$N_x\, N_p \geq \frac{\sigma_x^2\,\sigma_p^2}{|\gamma|}\;(\hbar/2)^2 \qquad (6)$$

in terms of entropy powers. This relation, like the Schrödinger-Robertson uncertainty relation eq. (2), is saturated by all pure Gaussian states and moreover, we can easily see that it, in fact, implies eq. (2).

We also prove an extended version of the above entropy-power uncertainty relation that is valid for n modes and is saturated for all n-mode Gaussian pure states. It can be expressed as

$$h(\hat{x}_1, ..., \hat{x}_n) + h(\hat{p}_1, ..., \hat{p}_n) - \frac{1}{2}\ln\left(\frac{|\gamma_x||\gamma_p|}{|\gamma|}\right) \geq n\,\ln(\pi e \hbar) \qquad (7)$$

where γ_x (γ_p) is the reduced covariance matrix of the x (p) quadratures.

Finally, we propose, as an extension of the work done by Huang [7], a n-modal version of entropic uncertainty relation expressing the balance between any two n-modal projective Gaussian measurements. Namely, we compute the differential entropies $h(\hat{A}_1, ...\hat{A}_n)$ and $h(\hat{B}_1, ...\hat{B}_n)$ where $(\hat{A}_1, ...\hat{A}_n)$ are the $\hat{x}$-quadratures measured after applying a Gaussian unitary A on the n-mode state $|\psi\rangle$ (and similarly for B). The bound of this uncertainty relation is expressed in terms of $\det(K)$ where $K_{ij} = [A_i, B_j]$ is the commutator between two quadratures.

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